1. Kétválasztható változójú differenciálegyenletek:

$$y'(x) = f(x)\, g(y(x))$$

$f : J \to \mathbb{R}$ folytonos

$g : J \to \mathbb{R} \setminus \{0\}$ folytonos

$$\left.\begin{array}{l} y'(x) = f(x)\, g(y(x)) \\ y(x_0) = y_0 \end{array}\right\} \text{Cauchy-feladat}$$

$$F(x) = \int_{x_0}^{x} f \quad , x \in J$$

$$G(y) = \int_{y_0}^{y} \frac{1}{g}$$

$\varphi : J_0 \to \mathbb{R}$ (1) megoldása

Ez azt jelenti, hogy φ' megoldása:

$$\varphi'(x) = f(x)\, g(\varphi(x)) \Rightarrow$$

$$\frac{\varphi'(x)}{g(\varphi(x))} = f(x) \quad , x \in J_0 \Rightarrow$$

$$G(\varphi(x))' = F'(x) \Rightarrow$$

$$\exists\, c \in \mathbb{R}, \text{ hogy } G(\varphi(x)) = F(x) + c \Rightarrow$$

$$\varphi(x) = G^{-1}(F(x) + c), \ x \in J_0$$

A Cauchy-feladat megoldása: $\varphi(x) = G^{-1}(F(x)),\ x \in J$

$$\frac{dy}{dx} = f(x)g(y)$$

$$\frac{dy}{g(y)} = f(x)\,dx$$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

$$y = \dots$$

Feladatok:

✓ 1) $y'(x) = y(x)$

✓ 2) $y'(x) = xy(x)$

✓ 3) $y'(x) = 4x\sqrt{y(x)}$, $y(1) = 1$

✓ 4) $y'(x) = xe^{y(x)}$

? 5) $y'(x) = (1+y^2(x))\ln x$

Pl.: $y'(x) = e^{y(x)-x} = e^{y(x)} \cdot e^{-x}$

$$\frac{dy}{dx} = e^{-x} \cdot e^{y} \qquad /:e^{y}$$

$$\int e^{-y}\,dy = \int e^{-x}\,dx$$

$$-e^{-y} = -e^{-x} - c$$

$$e^{-y} = e^{-x} + c$$

$$-y = \ln(e^{-x}+c)$$

$$\underline{y(x) = \ln(e^{-x}+c)}$$

II. Változóiban homogén differenciálegyenletek:

$$y'(x) = f\left(\frac{y(x)}{x}\right) \quad , \quad f: J \to \mathbb{R} \text{ folytonos}$$

$$u(x) = \frac{y(x)}{x}$$

$$y(x) = x \cdot u(x)$$

$$y'(x) = u(x) + xu'(x) = f(u(x))$$

$$u'(x) = \frac{1}{x}\left(f(u(x)) - u(x)\right)$$

Feladatok:

6) $y'(x) = \dfrac{y(x)}{y(x)+x}$

7) $y'(x) = \dfrac{y(x)+x}{x}$

8) $y'(x) = \dfrac{y(x)+x}{y(x)+3x}$

III. Elsőrendű lineáris differenciálegyenletek:

$f, g : J \to \mathbb{R}$ folytonos

(1) $y'(x) + f(x)y(x) = g(x)$

(2) $y'(x) + f(x)y(x) = 0$ $\to$ (1)-hez tartozó homogén lin. diffe.

φ_p (1) megoldása $\Leftrightarrow$ $c'(x)e^{-\int f(x)dx} + \underline{c(x)e^{-\int f(x)dx}} \cdot$

$\underbrace{\cdot(-f(x) + f(x)c(x)e^{-\int f(x)dx})}_{=0} = g(x) \Rightarrow$

$\Rightarrow c'(x) = g(x)e^{\int f(x)dx} \Leftrightarrow c(x) = \int g(x)e^{\int f(x)dx}dx$

$\varphi_p(x) = \left(\int g(x)e^{\int f(x)dx}dx\right)e^{-\int f(x)dx}$, $x \in J$ intervallumon

(1) megoldása.

Feladatok:

✓ 9) $y'(x) - xy(x) = x^3$

✓ 10) $y'(x) + y(x) = e^{-x}$

✓ 11) $y'(x) + \frac{2}{x}y(x) = 3$, $y(1) = 0$

✓ 12) $y'(x) = y(x) + x$

Pl.: $y'(x) + y(x) \cdot tg(x) = \sin 2x$ $\quad x \in]0; \frac{\pi}{2}[$

$y\left(\frac{\pi}{4}\right) = 1$

$y'(x) = y(x) \cdot tg(x) = 0$

$y(x) = c \cdot e^{-\int tg\,x\,dx} = c \cdot e^{\int -\frac{\sin x}{\cos x}dx} = c \cdot e^{\ln \cos x} = c \cdot \cos x$

$y_p(x) = c(x)\cos x$

$c'(x)\cos x = \sin 2x = 2\sin x \cos x$

$c'(x) = 2\sin x$

$c(x) = -2\cos x$

$y_p(x) = -2\cos^2 x \Rightarrow y(x) = \cos x - 2\cos^2 x$; $x \in]0; \frac{\pi}{2}[$

$1 = y\left(\frac{\pi}{4}\right) = c \cdot \frac{\sqrt{2}}{2} - 2 \cdot \frac{1}{2}$

$\frac{4}{\sqrt{2}} = c = 2\sqrt{2}$

$y(x) = 2\sqrt{2}\cos x - 2\cos^2 x$ $\quad x \in]0; \frac{\pi}{2}[$

IV. Bernoulli differenciálegyenletek:

$$y'(x) + f(x)\,y(x) = g(x)\,y(x)^{\alpha}, \quad f,g : J \to \mathbb{R} \text{ folyt.}, \ \alpha \neq 1.$$

$$u(x) = y(x)^{1-\alpha}$$

$$y(x) = u(x)^{\frac{1}{1-\alpha}}$$

$$\frac{1}{1-\alpha}\, u(x)^{\frac{1}{1-\alpha}-1} \cdot u'(x) + f(x)\,u(x)^{\frac{1}{1-\alpha}} = g(x)\,u(x)^{\frac{\alpha}{1-\alpha}}$$

elosztjuk ezzel az egyenletet

$$u'(x) + (1-\alpha)\,f(x)\,u(x) = g(x)(1-\alpha)$$

Feladatok:

✓ 13) $y'(x) + y(x) = -\frac{1}{y(x)}$

✓ 14) $y'(x) - y(x) = -(1+x)\,y(x)^2$

✓ 15) $y'(x) + y(x) + y(x)^2 = 0$

? 16) $y'(x) - y(x) = x \cdot y(x)^3$

V. Ricatti típusú differenciálegyenletek:

$$y'(x) = f(x)\,y(x)^2 + g(x)\,y(x) + h(x) \qquad f,g,h : J \to \mathbb{R} \text{ folyt.}$$

$y = u + v$ alakban

$$u' + \underline{\underline{v'}} = f(u^2 + 2uv + \underline{\underline{v^2}}) + g(u+v) + \underline{\underline{h}}$$

$$u'(x) = \big(g(x) + 2v(x)\cdot f(x)\big)\,u(x) + f(x)\cdot u(x)^2$$

Feladatok:

17.) $y'(x) + \frac{1}{x}\,y(x) + y(x)^2 = \frac{4}{x^2} \qquad v(x) = \frac{2}{x}$

18) $y'(x) + \frac{1}{3}\,y(x)^2 + \frac{2}{3}\cdot\frac{1}{x^2} = 0 \qquad v(x) = \frac{1}{x}$

19.) $y'(x) + 2y(x)\,e^{x} - y(x)^2 = e^{2x} - e^{x} \qquad v(x) = e^{x}$

VI. Egzakt differenciálegyenletek:

$J_1; J_2 \subset \mathbb{R}$

$P, Q : J_1 \times J_2 \to \mathbb{R}$ folyt. fgv. $Q \neq 0$.

Az (1) $P(x, y(x)) + Q(x; y(x))\, y'(x) = 0$

egzakt, ha van olyan $u : J_1 \times J_2 \to \mathbb{R}$, hogy

$\partial_1 u = P$ és $\partial_2 u = Q$. $\Rightarrow J_1 \times J_2 - u$.

Megoldás meghatározása:

A $\varphi : J \to \mathbb{R}$ megoldása az (1) egzakt diff. e-nek $\Leftrightarrow$

$\Leftrightarrow P(x, \varphi(x)) + Q(x, \varphi(x))\, \varphi'(x) = 0 \qquad x \in J$.

$$\partial_1 u(x, \varphi(x)) + \partial_2 u(x; \varphi(x)) \qquad \varphi'(x) = 0$$

$$\frac{d}{dx} u(x, \varphi(x)) = 0 \qquad x \in J$$

Ez akkor és csak akkor van, ha:

$$\exists c \in \mathbb{R} : u(x, \varphi(x)) = c, \quad x \in J$$

Ha P és Q folyt. diff-ható.

$\exists\, u \quad \partial_1 u = P \qquad \partial_2 u = Q \Rightarrow$ folyt. diff-ható

$$\underbrace{\partial_2 \partial_1 u = \partial_2 P \qquad \partial_1 \partial_2 u = \partial_1 Q}$$

$$\partial_2 P = \partial_1 Q$$

Ha a fgv kétszer diff-ható

$P(x, y) = x^2 + y^2 \Rightarrow \partial_2 P(x, y) = 2y$

$Q(x, y) = 2xy \Rightarrow \partial_1 Q(x, y) = 2y$

$$u(x, y) = \int_0^x t^2\,dt + \int_0^y 2xt\,dt = \left[\frac{t^3}{3}\right]_0^x + \left[x t^2\right]_0^{t=y}$$

$x = t \qquad y = 0$ -t írunk

$u(x, y) = c$

$$\frac{x^3}{3} - 0 + xy^2 - 0 = \frac{x^3}{3} + xy^2 = c \Rightarrow y(x) = \dots$$

Feladatok:

20) $x^2 + y(x)^2 + 2x\,y(x)\,y'(x) = 0$

21) $2x\,y(x) + 3\,y(x)^2 + \left(x^2 + 6x\,y(x) - 2y(x)\right) y'(x) = 0$

22) $2x + y(x) + \left(x - 2y(x)\right) y'(x) = 0$

VII. Hiányos másodrendű differenciálegyenletek:

Mj.: nem hiányosak: $y''(x) = f(x, y(x), y'(x))$

I. $y''(x) = f_1(x, y'(x))$

$u(x) = y'(x)$

$u'(x) = f_1(x, u(x))$

II. $y'' = f_2(y(x), y'(x))$

Hat. meg olyan „p" fgv-t, melyre $y'(x) = p(y(x))$ teljesül.

„p" meghatározása:

$$y''(x) = p'(y(x)) \cdot y'(x) = p'(y(x)) \cdot p(y(x))$$

$$p'(y(x)) \cdot p(y(x)) = f_2(y(x)) \cdot p(y(x))$$

$$p'(t)p(t) = f_2(t; p(t)) \qquad (t \in y \text{ értékkészletének})$$

Feladatok:

23) $2x y''(x) + y'(x) = 0$

24) $2y(x) y'(x) = y''(x)$

25) $y''(x) = \dfrac{y'(x)^2}{y(x)}$

26) $y''(x) = e^{y(x)}$

Pl.: Cauchy-feladat:

$$y''(x) = y'(x) e^{y(x)}$$
$$y(0) = 0 \quad ; \quad y'(0) = 1$$

$y'(x) = p(y(x))$ $\quad x = 0 \quad 1 = p(0)$

$p'(y(x))\, p(y(x)) = p(y(x))\, e^{y(x)}$

$p'(x) \cdot p(x) = p(x) \cdot e^x$

$p'(x) = e^x$

$p(x) = e^x + c \Rightarrow c = 0$

$p(x) = e^x$

$y'(x) = e^{y(x)}$

$\dfrac{dy}{dx} = e^y$

$\int e^{-y}\, dy = \int dx$

$-e^{-y} + c_1 = x + c_2$

$-e^{-y} = x + k$

$\hookrightarrow k = -1$ (felt. miatt)

$e^{-y} = 1 - x$

$y(x) = -\ln(1 - x)$