ZH-k: okt. 28. ; dec. 9.

1, táblázat : deriválási szabályok

2, alapfüggvények deriváltjai (elemi fgv-ek)

$f(x)$	$f'(x)$	
$\exp(x)$	$\exp x$	$f:\mathbb{R}\to\mathbb{R}$
$\log(x) = \ln x$	$\frac{1}{x}$	$f:\mathbb{R}^+\to\mathbb{R}$
$\exp_a x = a^x$	$a^x \ln a = (\exp_a x)\log a$	$f:\mathbb{R}\to\mathbb{R}$
$\log_a x$	$\frac{1}{x\log a} = \frac{1}{x\ln a}$	$f:\mathbb{R}^+\to\mathbb{R}$
x^a	$a\cdot x^{a-1}$	$f:\mathbb{R}^+\to\mathbb{R}$
$\sin x$	$\cos x$	$f:\mathbb{R}\to\mathbb{R}$
$\cos x$	$-\sin x$	$f:\mathbb{R}\to\mathbb{R}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$f:[-1;1]\to\mathbb{R}$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$	$f:[-1;1]\to\mathbb{R}$
$\operatorname{tg} x$	$\frac{1}{\cos^2 x}$	
$\operatorname{ctg} x$	$\frac{-1}{\sin^2 x}$	
$\operatorname{arctg} x$	$\frac{1}{1+x^2}$	
$\operatorname{arcctg} x$	$-\frac{1}{1+x^2}$	

$c\cdot f' = (c\cdot f)'$

$(f+g)' = f'+g'$

$(f\cdot g)' = f'\cdot g + f\cdot g'$

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$(f^{-1})'(y) = \frac{1}{f'(x)}$

$(f\circ g)'(x) = (f(g(x)))' = f'(g(x))\cdot g'(x)$

$(e^x)' = e^x$

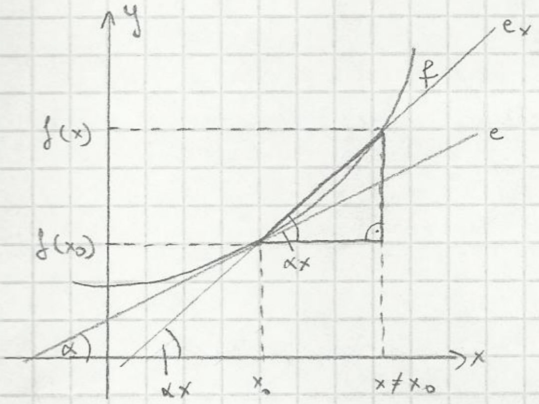
$(\ln x)' = \frac{1}{e^{\ln x}}$

$f(g(h(x)))' = f'(g(h(x)))\cdot g'(h(x))\cdot h'(x)$

IX. 23.

2. gyakorlat

Differenciálszámítás



$x \to x_0$ esetén „$e_x \to e$"

$\alpha_x \to \alpha$

$$\operatorname{tg} \alpha_x = \frac{f(x) - f(x_0)}{x - x_0} \to \operatorname{tg} \alpha$$

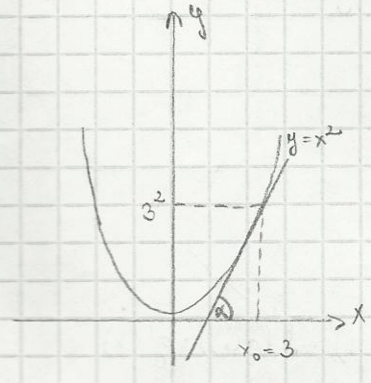
(mert a tg fgv folytonos)

$$\underbrace{\lim_{x \to x_0} \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\text{differenciahányados (ez egy fgv)}}}_{\text{differenciálhányados (ez egy szám)}} = \operatorname{tg}\alpha$$

(az érintő iránytényezője)

$= f'(x_0)$ (f-nek x_0-ban vett differenciálhányadosa)

Pl.:



$$\lim_{x \to 3} \frac{x^2 - 3^2}{x - 3} = \lim_{x \to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}} = \underline{\underline{6}}$$

$e: \ y = mx + b$

$9 = 6 \cdot 3 + b \ \to b = -9$

$e: \ \underline{\underline{y = 6x - 9}}$

Pl.:

$$\lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} \frac{\cancel{(x - x_0)}(x + x_0)}{\cancel{x - x_0}} = 2x_0$$

($\to$ az x^2 deriváltja az x_0-ban $2x_0$)

$(x^2)' = 2x$

Deriválási szabályok:

$(k \cdot f(x))' = k \cdot f'(x)$ (homogenitás)

$(f(x) \pm g(x))' = f'(x) \pm g'(x)$ (additivitás)

$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$$

$(f(g(x)))' = f'(g(x)) \cdot g'(x)$

Alapderiváltak:

$(\text{konstans})' = 0$

$x' = 1$

$(x^k)' = k \cdot x^{k-1}$ $(k \in \mathbb{R})$

$(a^x)' = a^x \cdot \ln a$ $(a>0)$

$(e^x)' = e^x$

$(\log_a x)' = \frac{1}{x \ln a}$

$(\ln x)' = \frac{1}{x}$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\operatorname{tg} x)' = \frac{1}{\cos^2 x} = 1 + \operatorname{tg}^2 x$

$(\operatorname{ctg} x)' = -\frac{1}{\sin^2 x} = -1 - \operatorname{ctg}^2 x$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

$(\operatorname{arctg} x)' = \frac{1}{1+x^2}$

$(\operatorname{arcctg} x)' = -\frac{1}{1+x^2}$

Példák:

$(x^{\frac{1}{2}})' = \frac{1}{2} x^{-\frac{1}{2}}$

1, $(5x^3)' = 5 \cdot 3 \cdot x^2 = 15x^2$

2, $(3\sin x + 5\operatorname{tg} x)' = 3\cos x - \frac{5}{\cos^2 x}$

3, $\left(\frac{\log_3 x}{\sqrt{x}}\right)' = \frac{(\log_3 x)'\sqrt{x} - (\sqrt{x})'(\log_3 x)}{x} = \frac{\frac{\sqrt{x}}{x \ln 3} - \frac{1}{2\sqrt{x}} \cdot \log_3 x}{x}$

4, $(\log_x (2x))' = \left(\frac{\ln(2x)}{\ln x}\right)' = \left(\frac{\ln 2 + \ln x}{\ln x}\right)' = \frac{\frac{1}{x} \cdot \ln x - (\ln 2 + \ln x) \cdot \frac{1}{x}}{\frac{1}{x^2}}$

5, $(\ln(2x))' = \frac{1}{2x} \cdot 2 = \frac{1}{x}$

6., $\left(\sqrt{3x^2+5}\right)' = \left((3x^2+5)^{\frac{1}{2}}\right)' = \frac{1}{2}(3x^2+5)^{-\frac{1}{2}} \cdot (6x)$

7., $\left(3^{\sin x}\right)' = 3^{\sin x} \cdot \ln 3 \cdot \cos x$

8., $\left(\frac{\operatorname{arctg}(\sin 2x)}{\sqrt{\log_3(\cos 3x)}}\right)' = \frac{(\operatorname{arctg}(\sin 2x))' \cdot (\log_3(\cos 3x))^{\frac{1}{2}} - \operatorname{arctg}(\sin 2x)\left(\sqrt{\log_3(\cos 3x)}\right)'}{\log_3(\cos 3x)}$

$(\operatorname{arctg}(\sin 2x))' = \frac{1}{1+(\sin 2x)^2} \cdot \cos 2x \cdot 2$

$\left(\sqrt{\log_3(\cos 3x)}\right)' = \frac{1}{2\sqrt{\log_3(\cos 3x)}} \cdot (\log_3(\cos 3x))'$

$(\log_3(\cos 3x))' = \frac{1}{\cos 3x \cdot \ln 3} \cdot (-\sin 3x) \cdot 3$

9., $(x^x)' = (e^{x\ln x})' = e^{x\ln x} \cdot \underbrace{(x \cdot \ln x)'}_{\ln x + 1} = x^x(\ln x + 1)$

10., $\left((\sin x)^{\cos x}\right)' = \left(\left(e^{\ln \sin x}\right)^{\cos x}\right)' = \left(e^{\cos x \ln \sin x}\right)' =$

$= e^{\cos x \cdot \ln \sin x} \cdot (\cos x \ln \sin x)' = \sin x^{\cos x} \cdot \left(-\sin x \ln \sin x + \cos \frac{1}{\sin x} \cos x\right)$

Hf.: $f'(x) = \left(\operatorname{tg} \sqrt[3]{6+3x^2}\right)' = \frac{1}{\cos^2(\sqrt[3]{6+3x^2})} \cdot \underbrace{\left(\sqrt[3]{6+3x^2}\right)'}_{\frac{1}{3}(6+3x^2)^{-\frac{2}{3}} \cdot 6x}$

$f'(x) = \left(\sqrt[5]{\cos^2(5x^3+3) + \sin 6x^7}\right)'$

$f'(x) = \frac{1}{5}\left(\cos^2(5x^3+3) + \sin 6x^7\right)^{-\frac{4}{5}} \cdot \left(\cos^2(5x^3+3) + \sin 6x^7\right)'$

$\cos^2(5x^3+3) + \sin 6x^7 = 2\cos(5x^3+3) \cdot (-\sin 5x^3+3) \cdot 15x^2 + \cos 6x^7 \cdot 42x^6$

$f'(x) = (\sin^4(5x^2))' = 4\sin(5x^2)^3 \cdot \cos 5x^2 \cdot 10x$

$f'(x) = \left(\frac{x+2}{x^2+1}\right)^5 = 5 \cdot \left(\frac{x+2}{x^2+1}\right)^4 \cdot \underbrace{\left(\frac{x+2}{x^2+1}\right)'}_{\frac{(x^2+1) - 2x(x+2)}{(x^2+1)^2}}$

$f'(x) = \left(\sqrt[5]{x} \cdot \sin(7 \cdot x^{\frac{4}{3}})\right)' = \left(x^{\frac{1}{5}} \cdot \sin(7 \cdot x^{\frac{4}{3}})\right)' =$

$= \frac{1}{5}x^{-\frac{4}{5}} \cdot \cos(7x^{\frac{4}{3}}) \cdot \frac{28}{3}x^{\frac{1}{3}}$

$$f'(x) = \left(\ln \frac{x^2+1}{x^4+3}\right)' = \frac{x^4+3}{x^2+1} \cdot \frac{2x(x^4+3) - (4x^{4-1})(x^2+1)}{(x^4+3)^2}$$

$$f'(x) = \left(\frac{x}{7+2^{\frac{1}{x}}}\right)' = \frac{7+2^{\frac{1}{x}} - \left(7+2^{\frac{1}{x}}\right)' x}{\left(7+2^{\frac{1}{x}}\right)^2}$$

$$\left(7+2^{\frac{1}{x}}\right)' = 2^{\frac{1}{x}} \cdot \ln 2 \cdot -\frac{1}{x^2}$$

$$f'(x) = \left(\arcsin \frac{4}{x^2}\right)' = \frac{1}{\sqrt{1-\left(\frac{4}{x^2}\right)^2}} \cdot \underbrace{\left(\left(\frac{2}{x}\right)^2\right)'}$$

$$2 \cdot \frac{2}{x} \cdot \underbrace{\left(\frac{2}{x}\right)'}_{-\frac{2}{x^2}} = -\frac{8}{x^3} = -8 \cdot x^{-3}$$

3. gyakorlat

IX. 30.

1,

$$\left(\sqrt[3]{\cos^2(\log_2 \sin x)}\right)' = \frac{2}{3}\left(\cos(\log_2 \sin x)\right)^{-\frac{1}{3}} \cdot \left(\cos(\log_2 \sin x)\right)' =$$

$$\left(\cos(\log_2 \sin x)\right)' = -\sin(\log_2 \sin x) \cdot \underbrace{(\log_2 \sin x)'}_{\frac{1}{\sin x \ln 2} \cdot \cos x}$$

2,

$$\left(3^{\sqrt{\frac{x^2+2}{\operatorname{arctg}(\sin^2 x)}}}\right)' = 3^{\sqrt{\frac{x^2+2}{\operatorname{arctg}(\sin^2 x)}}} \cdot \ln 3 \left(\sqrt{\frac{x^2+2}{\operatorname{arctg}(\sin^2 x)}}\right)'$$

$$\frac{1}{2}\left(\frac{x^2+2}{\operatorname{arctg}(\sin^2 x)}\right)^{-\frac{1}{2}} \cdot \left(\frac{x^2+2}{\operatorname{arctg}(\sin^2 x)}\right)'$$

$$\frac{2x(\operatorname{arctg}(\sin^2 x) - \frac{1}{1+(\sin^2 x)^2} \cdot (\sin^2 x)'}{(\operatorname{arctg}(\sin^2 x))^2}$$

$$(\sin^2 x)' = 2 \sin x \cdot \underbrace{(\sin x)'}_{\cos x}$$

$$y = e^{\ln y}$$

3.,

$$\left(\sqrt[\sin x]{3\log_4(2x)}\right)' = \left(3\log_4(2x)^{\frac{1}{\sin x}}\right)' =$$

$$=\left(e^{\ln(3\log_4(2x))^{\frac{1}{\sin x}}}\right)' = \left(e^{\frac{\ln(3\log_4(2x))}{\sin x}}\right)' =$$

$$= e^{\frac{\ln(3\log_4(2x))}{\sin x}} \cdot \left(\frac{\ln(3\log_4(2x))}{\sin x}\right)'$$

$$\left(\frac{\ln(3\log_4(2x))}{\sin x}\right)' = \frac{(\ln(3\log_4(2x)))'\cdot \sin x - \cos x(\ln(3\log_4(2x)))}{(\sin x)^2}$$

$$(\ln(3\log_4(2x)))' = \frac{1}{3\log_4(2x)}\cdot(3\log_4(2x))'$$

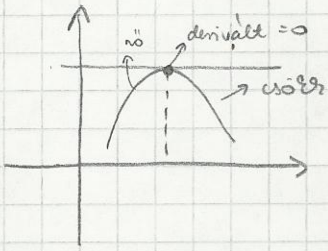
$$(3\log_4(2x))' = 3\cdot\frac{1}{2x\ln 4}\cdot\underbrace{(2x)'}_{2}$$

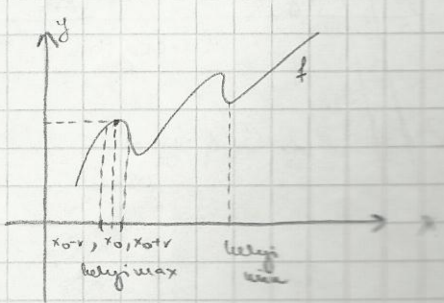
Teljes függvényvizsgálat

1., $f: (a,b) \to \mathbb{R}$, diff-ható, f' minden pontban > 0 $\Rightarrow$
$f'(x) > 0 \quad \forall x \in (a,b)$
$(<)$

$\Rightarrow$ f szig mon nő

(f szig mon csökk.)

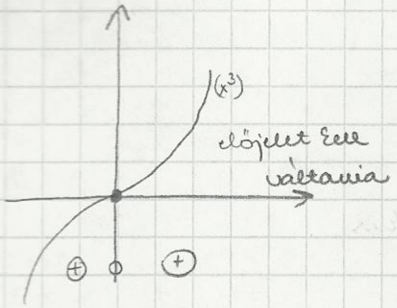


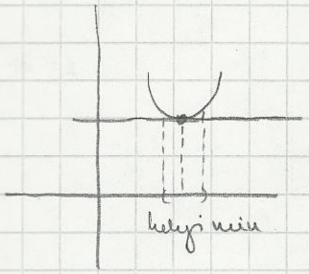


Def.: $H \to \mathbb{R}$, $(H \subset \mathbb{R})$ $x_0 \in H$ helyi maximuma (min) van, ha x_0-nak $\exists$ olyan $r > 0$ sugarú környezete, melyre teljesül, hogy $f(x_0) \overset{(\leq)}{\geq} f(x)$ $\forall x \in H \cap (x_0 - r, x_0 + r)$.

2., $f : (a,b) \to \mathbb{R}$, diff-ható, x_0-ban $x_0 \in (a,b)$ $f'(x) = 0$

$\exists\, r > 0$, hogy $f'(x) \overset{(<)}{>} 0$ $\forall x \in (a,b) \cap$ $(x_0 - r, x_0)$ és $f'(x) \overset{(>)}{<} 0$ $\forall x \in (a,b) \cap$ $(x_0, x_0 + r)$ $\Rightarrow$ f-nek x_0-ban helyi (min) maximuma van





konkáv

konvex

$f''(x) = (f'(x))'$

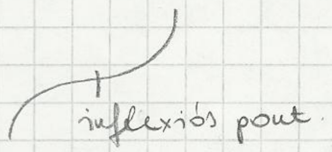
$f'''(x) = (f''(x))'$

$f^{(4)}(x) = (f'''(x))'$

$\vdots$

$f^{n}(x) = (f^{n-1}(x))'$

3., $f : (a,b) \to \mathbb{R}$, kétszer diff-ható fgv, $f''(x) \overset{(<)}{>} 0$ $\forall x \in (a,b) \Rightarrow f$ (a,b)-n konvex (konkáv).

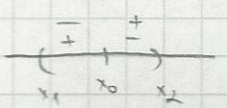


4., $f : (a,b) \to \mathbb{R}$, kétszer diff-ható, $x_0 \in (a,b)$, $f''(x_0) = 0$

(x_0 r sugarú környezete x_0-nak)
$\exists\, r > 0$, hogy $f''(x_1) \cdot f''(x_2) < 0$ $\forall x_1 \in (a,b) \cap (x_0 - r, x_0)$

$\underbrace{\forall x_2 \in (a,b) \cap (x_0, x_0 + r)}$

f-nek x_0-ban $\exists$ inflexiós pontja



Feladat: $f: \mathbb{R} \to \mathbb{R}$ $f(x) := \dfrac{10x}{1+x^2}$

I, (ha nincs megadva ért. tart. azt is meg kell adni)

ZÉRUSHELY:

$$\frac{10x}{1+x^2} = 0 \qquad 10x = 0 \Rightarrow \underline{\underline{x=0}}$$

II. **PARITÁS:**

páros: ha az y tengelyre szimmetrikus
páratlan: ha az origóra —"—

$$f(-x) = \frac{10(-x)}{1+(-x^2)} = \frac{-10x}{1+x^2} = -\frac{10x}{1+x^2} = -f(x)$$

A fgv páratlan.

III. **HATÁRÉRTÉK:**

Hol kell határértéket nézni?

a, $+\infty$, $-\infty$-ben, ha az ért. tart. megengedi

b, szakadási helyeken jobb és baloldali hé-t kell nézni

c, az ért. tart szélein

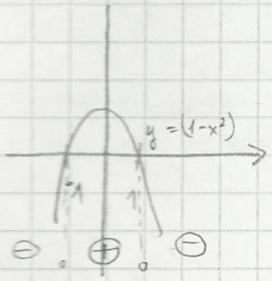
$+\infty$-ben

$$\lim_{x\to\infty} \frac{10x}{1+x^2} = \frac{10\overbrace{\left(\frac{1}{x}\right)}^{\to 0}}{\underbrace{\frac{1}{x^2}}_{0}+1} = \frac{0}{1} = 0$$

$$\lim_{x\to -\infty} \frac{10x}{1+x^2} = 0$$

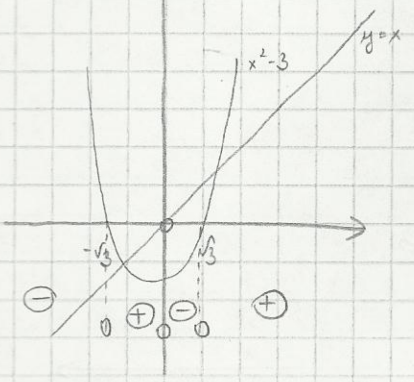
IV. $f'(x) = \left(\dfrac{10x}{1+x^2}\right)' = \dfrac{10(1+x^2)-(2x)\cdot 10x}{(1+x^2)^2} = \dfrac{10+10x^2-20x^2}{(1+x^2)^2} = \dfrac{10-10x^2}{(1+x^2)^2} =$

$$= \underbrace{\frac{10}{(1+x^2)^2}}_{\substack{+ \\ \text{mindig} > 0}} (1-x^2)$$



V. $$f''(x)=\left(\frac{10-10x^2}{(1+x^2)^2}\right)'=\left(10\,\frac{1-x^2}{(1+x^2)^2}\right)'=10\,\frac{-2x\cdot(1+x^2)^{\cancel{2}}-(1-x^2)\,2\,\cancel{(1+x^2)}\,2x}{(1+x^2)^{\cancel{4}\,3}}=$$

$$=20\,\frac{-1-x^2-2+2x^2}{(1+x^2)^3}=20x\,\frac{x^2-3}{\underbrace{(1+x^2)^3}_{+}}\cdot x(x^2-3)$$



		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$		
f'	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	
f''	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	
f	↘	↘	↘	helyi min $y=-5$	↗	↗	↗	helyi max $y=5$	↘	↘	↘	
		infl. p. $-2{,}5\sqrt{3}$				infl. p. 0				infl. p. $2{,}5\sqrt{3}$