x.14.

1. gyakorlat

Ő a zsebkendőjás ebben a módban!

$$f(x) = \frac{x^3}{3-x^2}$$

1, $D_f = \mathbb{R} \setminus \{\sqrt{3}, -\sqrt{3}\}$ (ért. tartomány)

2, zérushely megadása

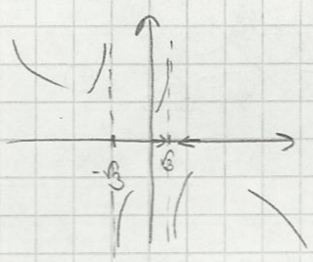
$$x^3 = 0 \Rightarrow \underline{\underline{x = 0}}$$

3, paritás vizsgálat

$$f(-x) = \frac{(-x)^3}{3-(-x)^2} = \frac{-x^3}{3-x^2} = -f(x) \qquad \text{páratlan}$$

4., határértékek kiszámolása

$$\lim_{x \to \infty} \frac{x^3}{3-x^2} = \lim_{x \to \infty} \frac{\overset{\to\infty}{x}}{\underbrace{\frac{3}{x^2}}_{\to 0} - 1} = \frac{\infty}{-1} = -\infty$$



$$\lim_{x \to -\infty} \frac{x^3}{3-x^2} = \frac{-\infty}{-1} = \infty$$

Való. hé.: $\lim\limits_{x \to \sqrt{3}-0} \frac{x^3}{3-x^2} = \lim\limits_{x \to \sqrt{3}-0} \underbrace{\frac{x^3}{\sqrt{3}+x}}_{\to \frac{\sqrt{3}^3}{2\sqrt{3}}} \cdot \frac{1}{\sqrt{3}-x} \to \infty$

$$x \to \sqrt{3} \qquad x < \sqrt{3}$$

$$\sqrt{3} - x \to 0 \qquad 0 < \sqrt{3} - x$$

! ⊕ tagú nullsorozat reciproka VÉGTELENBE TART

$$\frac{1}{\frac{1}{n}} = n \to \infty$$

$$\lim_{x \to \sqrt{3}+0} \frac{x^3}{3-x^2} = \lim_{x \to \sqrt{3}+0} \underbrace{\frac{x^3}{\sqrt{3}+x}}_{\to \frac{\sqrt{3}^3}{2\sqrt{3}}} \cdot \frac{1}{\sqrt{3}-x} \to -\infty$$

$$x \to \sqrt{3} \qquad x > \sqrt{3}$$

$$\sqrt{3} - x \to 0 \qquad \sqrt{3} - x < 0$$

! Negatív tagú nullsorozat reciproka

– VÉGTELENBE TART

$$\lim_{x\to -\sqrt{3}-0} \frac{x^3}{3-x^2} = \lim_{x\to -\sqrt{3}-0} \frac{x^3}{\sqrt{3}-x}\cdot\frac{1}{\sqrt{3}+x} \to -\infty = \infty$$

$$\frac{x^3}{\sqrt{3}-x} \to \frac{(-\sqrt{3})^3}{2\sqrt{3}} = -\frac{3}{2} < 0$$

$x \to -\sqrt{3}$ $\qquad$ $x < -\sqrt{3}$

$\sqrt{3}+x \to 0$ $\qquad$ $\sqrt{3}+x < 0$

$\Downarrow$

$$\frac{1}{\sqrt{3}+x} \to -\infty$$

$$\lim_{x\to -\sqrt{3}+0} \frac{x^3}{3-x^2} = \lim_{x\to -\sqrt{3}+0} \frac{x^3}{\sqrt{3}-x}\cdot\frac{1}{\sqrt{3}+x} \to -\infty$$

$$\frac{x^3}{\sqrt{3}-x} \to -\frac{3}{2} < 0$$

$x \to -\sqrt{3}$ $\qquad$ $x > -\sqrt{3}$

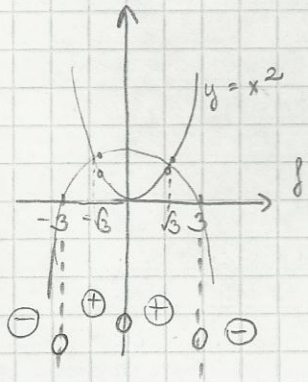
$\sqrt{3}+x \to 0$ $\qquad$ $\sqrt{3}+x > 0$

$$\frac{1}{\sqrt{3}+x} \to \infty$$

5; $$f'(x) = \left(\frac{x^3}{3-x^2}\right)' = \frac{3x^2\cdot(3-x^2) - (3-x^2)'\cdot x^3}{(3-x^2)^2} = \frac{3x^2(3-x^2) - x^3\cdot(-2x)}{(3-x^2)^2} =$$

$$= \frac{9x^2 - 3x^4 + 2x^4}{(3-x^2)^2} = \frac{9x^2 - x^4}{(3-x^2)^2} = \frac{x^2(9-x^2)}{(3-x^2)^2}$$

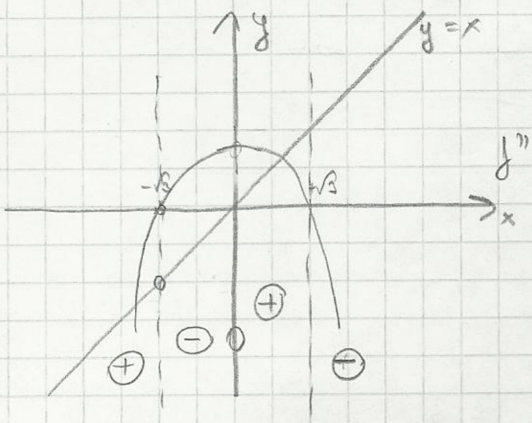
$\to$ Ez mindig $\oplus$



6; $$f''(x) = \left(\frac{9x^2-x^4}{(3-x^2)^2}\right)' = \frac{(18x-4x^3)\cancel{(3-x^2)^2} - (9x^2-x^4)\cdot 2\cdot\cancel{(3-x^2)}\cdot(-2x)}{(3-x^2)^{\cancel{4}\,3}} =$$

$$= \frac{54x - 18x^3 - 12x^3 + \cancel{4x^5} + 36x^3 - \cancel{4x^5}}{(3-x^2)^3} = \frac{6x^3+54x}{(3-x^2)^3} = \frac{6x(x^2+9)}{(3-x^2)^3}$$

$(x^2+9) \to \oplus$

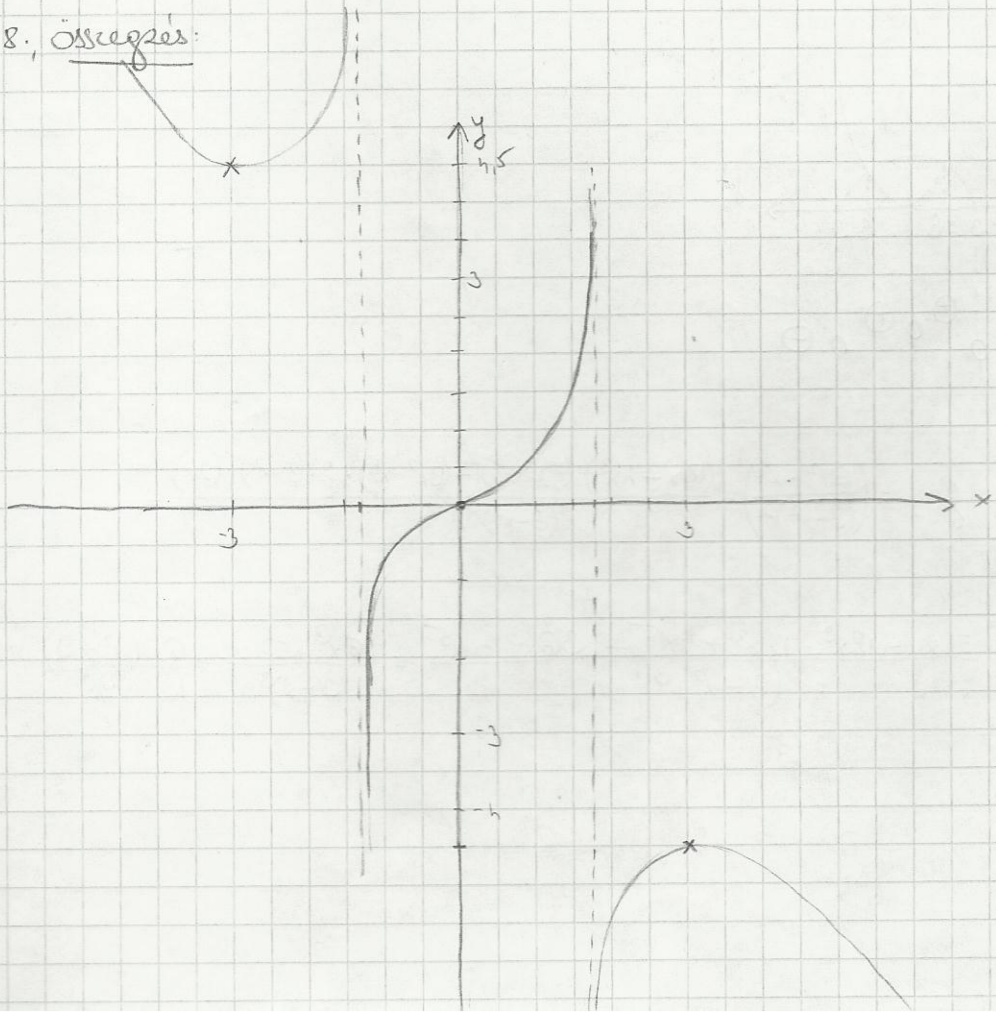


7., Táblázat:

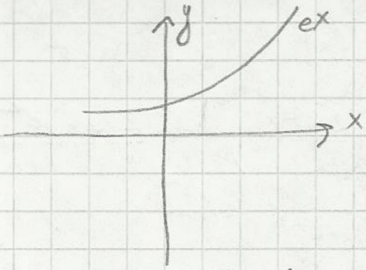
		-3		$-\sqrt{3}$		0		$\sqrt{3}$		3	
f'	−	0	+		+	0	+		+	0	−
f''	+	+	+		−	0	+		−	−	−
f	↘	helyi min. $x = 4{,}5$	↗		↗	0	↗		↗	max. $-4{,}5$	↘
f	∪	∪	∪		∩	infl. 0	∪		∩	∩	∩

$$f(-3) = \frac{(-3)^3}{3-(-3)^2} = \frac{-27}{-6} = 4{,}5$$

8., Összegzés:



$$f(x) = \frac{x^2}{e^x}$$



1, $D_f = \mathbb{R}$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

2, zh: $x^2 = 0$, $x = 0$

3, paritás

$$f(-x) = \frac{(-x)^2}{e^{-x}} = x^2 \cdot e^x \neq \begin{matrix} f(x) \\ -f(x) \end{matrix}$$

nincs paritás

4, határértékek

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \frac{(2x)'}{(e^x)'} = \frac{2}{e^x} = 0.$$

L'Hospital szabály

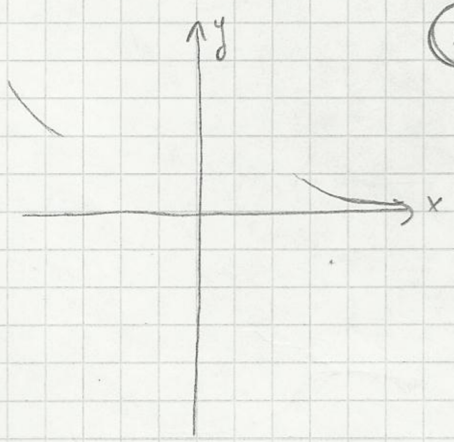
Nem a fgv deriváltja kell a határértékhez,

külön kell deriválni a sz-t és a n-t.

Ha egy számot osztunk $\pm\infty$-nel $\Rightarrow$ az 0.

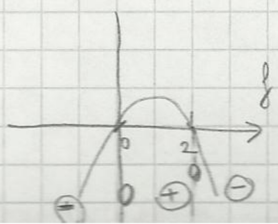
$$\lim_{x\to-\infty} \frac{x^2}{e^x} = \lim_{x\to-\infty} x^2 e^{-x} = \infty$$

hé



5,

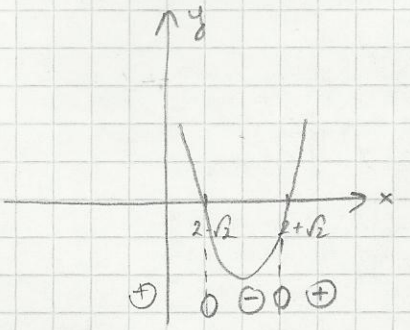
$$f'(x) = \left(\frac{x^2}{e^x}\right)' = \frac{2x \cdot e^x - x^2 \cdot e^x}{(e^x)^2} = \frac{2x - x^2}{e^x} = \frac{x(2-x)}{e^x}$$



6., $f''(x)$

$$\left(\frac{2x-x^2}{e^x}\right)' = \frac{(2-2x)e^x - e^x(2x-x^2)}{(e^x)^2} = \frac{2-2x-2x+x^2}{e^x} =$$

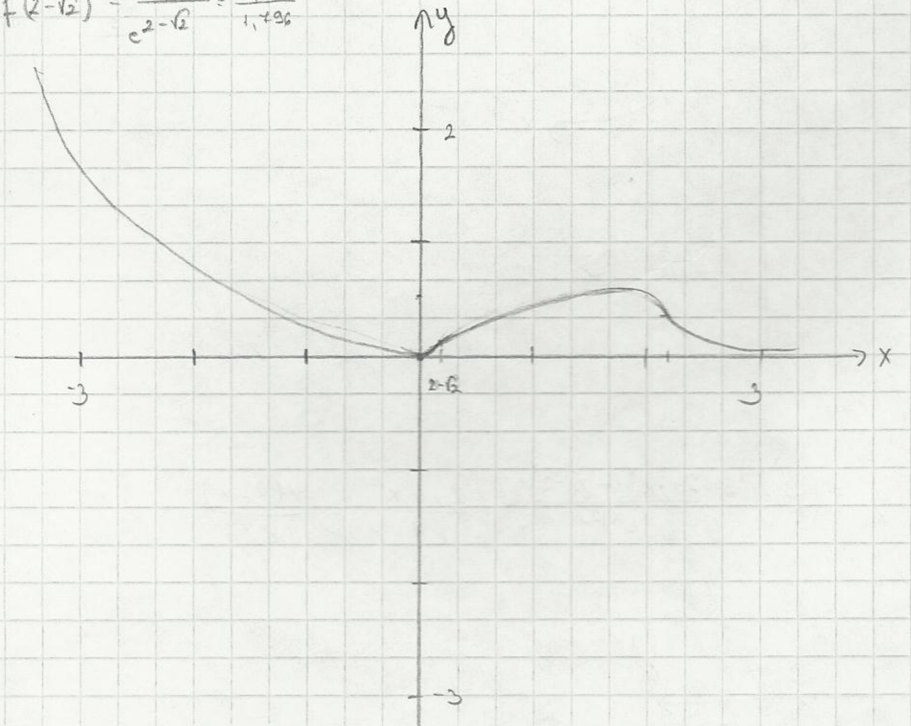
$$= \frac{x^2-4x+2}{e^x} \qquad x_{1,2} = 2 \pm \sqrt{2}$$



$0;\ 2-\sqrt{2};\ 2;\ 2+\sqrt{2}$

		0		$2-\sqrt{2}$		2		$2+\sqrt{2}$	
f'	−	0	+	+	+	0	−	−	−
f''	+	+	+	0	−	−	−	0	+
f	↘	min 0	↗	↗	↗	max 0,541	↘	↘	↘
	⌣	⌣	⌣	inf. p. 0,19	⌢	⌢	⌢	inf. 0,38	⌣

$f(2-\sqrt{2}) = \frac{(2-\sqrt{2})^2}{e^{2-\sqrt{2}}} = \frac{0,343}{1,796}$



L'Hospital-szabály

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

és $\begin{cases} \text{ha } f(x) \to \pm\infty \\ \text{ha } g(x) \to \pm\infty \end{cases}$ féloldali határértékre is igaz

vagy $x_0 \in \underbrace{\mathbb{R} \cup \{+\infty; -\infty\}}_{\mathbb{R}_b}$

és $\begin{cases} f(x) \to 0 \\ g(x) \to 0 \end{cases}$

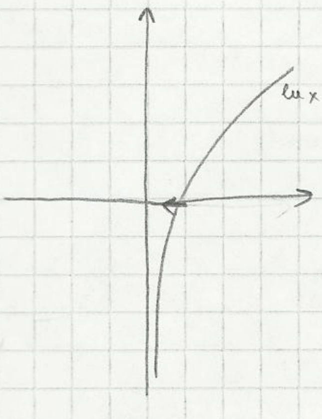
Pl.: $\lim\limits_{x\to 0+0} x \cdot \ln x = \lim\limits_{x\to 0+0} \dfrac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to 0+0} \dfrac{(\ln x)'}{(\frac{1}{x})'} = \dfrac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to 0+0} \dfrac{1}{x} \cdot \dfrac{-x^2}{1} =$

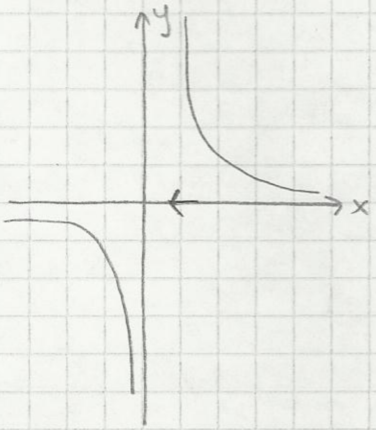
ln csak ⊕ x-ekre van értelmezve

nem veszi fel a 0 értéket

L'Hospital -ból

$= \lim\limits_{x\to 0+0} (-x) = 0$





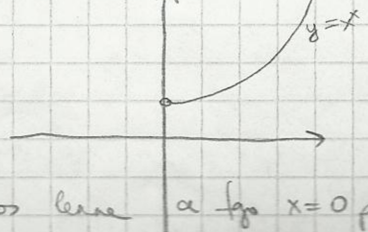
2; 0^0 nincs értelmezve!

analízis : $0^0 = 1$ (határátmenet miatt)

$\lim\limits_{x\to 0+0} x^x = \lim\limits_{x\to 0+0} (e^{\ln x})^x = \lim\limits_{x\to 0+0} e^{x \ln x} \overset{\text{lim}}{=} e^0 = 1$

előző alapján

$x = e^{\ln x}$ $(x>0)$



ha a 0^0 értelmezését, folytonos lenne a fgv $x=0$ pontban

Hf.: $\lim\limits_{x\to 0} \frac{x-\sin x}{x^3}$

$\lim\limits_{x\to 0} \frac{x^2 \cdot \sin 2x}{x-\sin x}$

$\lim\limits_{x\to 0} \frac{e^x - e^{-x}}{\sin x \cdot \cos x}$

(ZH) 5. gyakorlaton

6. gyakorlat

Integrálszámítás

$(x^2)' = 2x$ $\qquad$ $(x^2+5)' = 2x$

$2x$

Primitív fgv.

$F' = f$ $\Rightarrow$ a f-nek a F a primitív fgv-e.

- Ha F és G primitív fgv-ei primitív fgvei f-nek, akkor $F - G$ = konstans.

- Ha F primitív fgv-e f-nek $\Rightarrow$ a köv. jelölést használjuk.

$$\int f(x)\,d\underset{\text{szerint integrálunk}}{x} = F(x) + c \qquad (c \in \mathbb{R})$$

A 0-t nem fogjuk külön integrálni, a zem-vel nem foglalkozunk!

Integrálási szabályok!

$$\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\int k \cdot f(x)\,dx = k \int f(x)\,dx$$

I. Alapintegrálok: I. módszer

- $\int k\,dx = kx + c$ $\qquad (k, c \in \mathbb{R})$

$$\left\langle (x^k)' = k x^{k-1},\ \alpha := k-1,\ (x^{\alpha+1})' = (\alpha+1)\,x^{\alpha} \right.$$

$$\left. x^{\alpha} = \left(\frac{x^{\alpha+1}}{\alpha+1}\right)' = \left(\frac{x^{\alpha+1}}{\alpha+1}\right)' \right\rangle$$

- $\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C$ $(\alpha \neq -1;\ C \in \mathbb{R})$
- $\int \frac{1}{x} dx = \ln|x| + C$ $(C \in \mathbb{R})$
- $\int a^{x} dx = \frac{a^{x}}{\ln a} + C$ $(a>0, a \neq 1, C \in \mathbb{R})$
- $\int e^{x} dx = e^{x} + C$ $(C \in \mathbb{R})$
- $\int \sin x\, dx = -\cos x + C$ $(C \in \mathbb{R})$
- $\int \cos x\, dx = \sin x + C$ $(C \in \mathbb{R})$
- $\int \frac{1}{\cos^2 x} dx = \operatorname{tg} x + C$ $(C \in \mathbb{R})$
- $\int \frac{1}{\sin^2 x} dx = -\operatorname{ctg} x + C$ $(C \in \mathbb{R})$
- $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$ $(C \in \mathbb{R})$
- $\int \frac{1}{1+x^2} dx = \operatorname{arctg} x + C$ $(C \in \mathbb{R})$

<u>Példa:</u>

1; $\int \frac{3x^2 - 4\sqrt{x} + 2}{\sqrt{x}} dx = \int \left(3x^{2-\frac{1}{2}} - 4 + 2x^{-\frac{1}{2}}\right) dx =$

$= \int \left(3x^{\frac{3}{2}} - 4 + 2x^{-\frac{1}{2}}\right) dx = \int 3x^{\frac{3}{2}} dx + \int -4 dx + \int 2x^{-\frac{1}{2}} dx =$

$= 3\int x^{\frac{3}{2}} dx + \int (-4)\, dx + 2\int x^{-\frac{1}{2}} dx =$

$$= 3\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} - 4x + 2\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

TEVE-rab.

2; $\int \frac{x^2}{1+x^2}\,dx = \int \frac{(x^2+1)-1}{1+x^2}\,dx = \int \left(1 - \frac{1}{1+x^2}\right) dx =$

$= x - \operatorname{arctg} x + c$

3, $\int \left(1 + e^{x-1}\right) dx = \int \left(1 + \frac{1}{e}\cdot e^{x}\right) dx = x + \frac{1}{e} e^{x} + c$

$\int \frac{1}{x}\,dx = \ln|x| + c$

$\Downarrow$

$(\ln|x| + c)' = \frac{1}{x}$

$(\ln|f(x)| + c)' = \frac{1}{f(x)} \cdot f'(x)$

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c}$$ II. módszer

4, $\int \operatorname{ctg} x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{(\sin x)'}{\sin x}\,dx = \ln|\sin x| + c$

5, $\int \operatorname{tg} x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx =$

$= -\int \frac{(\cos x)'}{\cos x}\,dx = -\ln|\cos x| + c$

6., $\int \frac{1}{x \cdot \ln x}\,dx = \int \frac{\frac{1}{x}}{\ln x}\,dx = \int \frac{(\ln x)'}{\ln x}\,dx =$

$= \ln|\ln x| + c$

7, $\int \frac{1-\cos x}{1+\cos x}\,dx = \int \frac{(1-\cos x)^2}{(1+\cos x)(1-\cos x)}\,dx = \int \frac{(1-\cos x)^2}{\sin^2 x}\,dx =$

$= \int \frac{1 - 2\cos x + \cos^2 x}{\sin^2 x}\,dx = \int \frac{1 - 2\cos x + 1 - \sin^2 x}{\sin^2 x}\,dx =$

– [illegible] 108 az ágy... minthe meg is dobálta volna, ezt honnan veszi?

– Valószínű [illegible] a feladatot.

$$= \int \frac{2 - 2\cos x - \sin^2 x}{\sin^2 x}\, dx = \int \left(2 \cdot \frac{1}{\sin^2 x} - 2\frac{\cos x}{\sin^2 x} - 1\right) dx = (*)$$

$$< \left(\frac{x^{\alpha+1}}{\alpha+1} + C\right)' = x^{\alpha}$$

$$\left(\frac{f^{\alpha+1}(x)}{\alpha+1} + C\right)' = f^{\alpha}(x) \cdot f'(x) \Leftrightarrow \boxed{\int f'(x)\, f^{\alpha}(x)\, dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + C}$$

III. wzór

$$(\alpha \neq -1) \quad >$$

$$(*) = -2\operatorname{ctg} x - 2\int \Big(\underbrace{\cos x}_{f'(x)} \underbrace{(\sin x)^{-2}}_{f}\Big)\, dx - x =$$

$$= -2\operatorname{ctg} x - 2\frac{(\sin x)^{-2+1}}{-2+1} - x + C$$

8) $$\int \operatorname{tg}^2 x\, dx = \int \left(\frac{\sin x}{\cos x}\right)^2 dx = \int \frac{\sin^2 x}{\cos^2 x}\, dx =$$

$$= \int \frac{1 - \cos^2 x}{\cos^2 x}\, dx = \int \left(\frac{1}{\cos^2 x} - 1\right) dx =$$

$$= \operatorname{tg} x - x + C$$

I. wzór

9) $$\int \frac{1 + x + x^2}{x + x^3}\, dx = \int \frac{1 + x + x^2}{x(1+x^2)}\, dx = \int \left(\frac{\cancel{x}}{\cancel{x}(1+x^2)} + \frac{1 + x^2}{x(1+x^2)}\right) dx$$

$$= \int \left(\frac{1}{1+x^2} + \frac{1}{x}\right) dx = \underline{\operatorname{arctg} x + \ln|x| + C}$$

I.

10) $$\int \cos^3 x\, dx = \int \cos x \cdot \cos^2 x\, dx = \int \cos x\,(1 - \sin^2 x)\, dx =$$

$$= \int \Big(\underbrace{\cos x}_{\text{I.}} - \underbrace{(\sin x)' \cdot (\sin x)^2}_{\text{III.}}\Big)\, dx = \sin x - \frac{\sin^{2+1} x}{2+1} + C$$